Week 8
MATH 34B
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Given that there's a midterm this coming Friday based on the material from homeworks 5+6, I decided to compile some of the harder and less done questions from these homeworks. In this packet, there are two question from homework 5 and three from homework 6.
Here's how today's going to work. As usual, I will devote the first thirty minutes to letting you guys have a crack at these questions. Afterwards, I will present some of these problems. The main difference between today's section and the other sections we've had is that I will not be coming around to help, and there will be no collaboration. The intent of this is to simulate the conditions of an exam (where there will be neither collaboration allowed nor TAs walking you through problems). One thing I want to say is that, **I DO NOT CLAIM THIS WILL BE SIMILAR IN ANY WAY, SHAPE, OR FORM, TO THE ACTUAL MIDTERM.** In particular, I do not have any part in writing the midterm, nor did the professor have any part in the creation/compilation of this worksheet.

5.21 Find y in terms of x if $\frac{dy}{dx} = x^6 y^{-7}$ and $y(0) = 3$. For what x-interval is the solution defined?

$$\int y^7\, dy = \int x^6\, dx$$

$$\frac{y^8}{8} = \frac{x^7}{7} + C$$

Know: $y(0) = 3$.

$$\frac{3^8}{8} = \frac{0^7}{7} + C$$

$$\Rightarrow \frac{y^8}{8} = \frac{x^7}{7} + \frac{3^8}{8}$$

$$y^8 = 8\left(\frac{x^7}{7} + \frac{3^8}{8}\right)$$

$$y = 8^{1/8}\left(\frac{x^7}{7} + \frac{3^8}{8}\right)^{1/8}$$

For y to be defined, must have

$$\frac{x^7}{7} + \frac{3^8}{8} \geq 0 \iff \frac{x^7}{7} \geq \frac{-3^8}{8}$$

$$\iff x^7 \geq \frac{-7 \cdot 3^8}{8}$$

$$x \geq \left(\frac{-7 \cdot 3^8}{8}\right)^{1/7}$$

5.22 Let $Q(t)$ represent the amount of a certain reactant present at time t. Suppose that the rate of decrease of $Q(t)$ is proportional to $Q^3(t)$. That is, $Q' = -kQ^3$, where k is a positive constant of proportionality.

(a) Suppose $Q(0) = \frac{1}{4}$. How long will it take for the reactant to be reduced to one half of its original amount?

(b) Suppose $Q(0) = \frac{1}{2}$. How long will it take for the reactant to be reduced to one half of its original amount?

(c) Recall that, in problems of radioactive decay where the differential equation has the form $Q' = -kQ$, the half-life was independent of the amount of the amount of radioactive material initially present. What happens in the case of $Q' = -kQ^3$? Does half-life depend on $Q(0)$, the amount initially present?

$$\frac{dQ}{dt} = -kQ^3$$

$$\frac{dQ}{Q^3} = -k\,dt$$

$$-\frac{1}{Q^2} \cdot \frac{1}{2} = -kt + C.$$

a) $Q(0) = 1/4 \Rightarrow -\frac{1}{(1/4)^2} \cdot \frac{1}{2} = -k(0) + C \Rightarrow C = -8$

$$-\frac{1}{Q^2} \cdot \frac{1}{2} = -kt - 8$$

half of $1/4 = 1/8$

$$-\frac{1}{(1/8)^2} \cdot \frac{1}{2} = -kt - 8$$

$$\Rightarrow t = \frac{24}{k}$$

6.13 Solve the equation $y' = 15 - 6y$ with the initial condition $y'(0) = 0$

$$\frac{dy}{dt} = 15 - 6y = \frac{dy}{15-6y} = dt$$

$$-\frac{1}{6}\ln|15-6y| = t + C$$

$$\ln|15-6y| = -6t + C$$

$$15 - 6y = Ce^{-6t}$$

$$y = \frac{15}{6} + Ce^{-6t}.$$

$$y' = -6Ce^{-6t}$$

$$y'(0) = 0$$

$$C = 0$$

$$y = \frac{15}{6}$$

6.16 The terminal velocity of a person falling through the air is about $400km/hr$. The gravitational acceleration is $10ms^{-2}$. Use this information to find C in the equation $v' = C(\frac{g}{C} - v)$, where g is gravitational acceleration and v is velocity.

(a) How long does it take for a falling person to reach the 90 percent of this terminal velocity?

(b) How long to reach 95 percent?

(c) How long to reach 99 percent?

First, we notice $400 km/hr = \frac{400}{3.6} m/s$.

Also, have $\lim_{t\to\infty} v(t) = 400/3.6$ (since that is terminal velocity)

and $\lim_{t\to\infty} v'(t) = 0$, since v asymptotically approaches $400/3.6$.

So, taking $t \to \infty$, we have $0 = C\left(\frac{10}{C} - \frac{400}{3.6}\right)$,

so $C = \frac{36}{400}$

So, $\frac{dv}{dt} = \frac{36}{400}\left(\frac{10}{\left(\frac{36}{400}\right)} - v\right)$

$$\int \frac{dv}{\left(\frac{10}{\left(\frac{36}{400}\right)} - v\right)} = \int \frac{36}{400}\, dt$$

$$-\ln\left|\frac{10}{\left(\frac{36}{400}\right)} - v\right| = \frac{36}{400}t + C$$

$$\ln\left|\frac{10}{\left(\frac{36}{400}\right)} - v\right| = -\frac{36}{400}t + C$$

$$\frac{10}{\left(\frac{36}{400}\right)} - v = Ce^{-\frac{36}{400}t}$$

$$v = \frac{10}{\left(\frac{36}{400}\right)} - Ce^{-\frac{36}{400}t}$$

$v(0) = 0 \Rightarrow C = \frac{10}{\left(\frac{36}{400}\right)}$

$$v = \frac{10}{\left(\frac{36}{400}\right)} - \frac{10}{\left(\frac{36}{400}\right)} e^{-36/400\, t}$$

a) plug in $v = \frac{9}{10} \cdot \frac{400}{3.6} = 100$, solve for t...

$t = 100\log(10)/9$

b) plug in $\frac{95}{100} \cdot \frac{400}{3.6}$

$t = \frac{100 \log(20)}{9}$

c) .... $t = \frac{200\log(10)}{9}$

6.31 Find the solution to the differential equation $\frac{dz}{dt} = 8te^{7z}$ that passes through the origin.

(a) Sketch on graph paper a graph of the speed over a 3 second time span.

(b) What volume of blood passes along the artery in one second?

$$\int \frac{dz}{e^{7z}} = \int 8t\,dt$$

$$\frac{-e^{-7z}}{7} = 4t^2 + C, \text{ have } (0,0), \text{ so}$$

$$\frac{-e^{-7\cdot 0}}{7} = 4\cdot 0^2 + C$$

$$C = -\tfrac{1}{7}$$

$$\frac{e^{-7z}}{7} = 4t^2 + \left(-\tfrac{1}{7}\right)$$

$$e^{-7z} = -28t^2 + 1$$

$$-7z = \ln(1 - 28t^2)$$

$$z = -\tfrac{1}{7}\ln(1-28t^2)$$